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On Krein's Theorem for Indeterminacy of the Classical Moment Problem

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Krein's sufficient condition for indeterminacy states that a positive measure on the real line, having moments of all orders, is indeterminate provided it has density with respect to Lebesgue measure and that this density has a finite logarithmic integral. We generalize this result and we also give a discrete analogue.

1. INTRODUCTION

This exposition deals with sufficient conditions for indeterminacy of the classical Hamburger moment problem. We denote by $\mathscr{M}$ the set of probability measures μ on the real line having moments of all orders (meaning that any polynomial is integrable w.r.t. μ).

A measure $\mu \in \mathscr{M}$ is called indeterminate (resp. determinate) if there are other (resp. no other) measures in $\mathscr{M}$ having the same moments as μ. It is a classical fact that the polynomials form a dense subspace in $L^2(\mu)$ provided μ is determinate, see [1, p. 45]. Turning this around we thus have a sufficient condition for indeterminacy. We remark that the condition is not necessary: there exist indeterminate measures μ for which the polynomials do form a dense subspace of $L^2(\mu)$. Such a measure is called Nevanlinna-extremal and it is a discrete measure supported on the zero set of an entire function of at most exponential type zero, see [1, pp. 100–101]. An important issue in the classical moment problem is that of density of polynomials in various L^p-spaces, see, e.g., [5, 6]. It is closely related to weighted approximation over (subsets of) the real line. We mention the two papers [2, 11] and also [9, Chaps. VI and VIII, Sect. A].

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In 1945 Krein found the following result:

THEOREM 1.1. *Suppose that* $d\mu(t) = h(t)\, dt \in \mathcal{M}$. *If*

$$\int \frac{\log h(t)}{t^2+1}\, dt > -\infty$$

then μ *is indeterminate.*

Remark 1.2. An integral without limits is to be understood as an integral over the whole real line. A sum without indices denotes a sum over the integers.

In [10] it was obtained that $\operatorname{span}\{e^{i\alpha t} \mid \alpha \geqslant 0\}$ is dense in $L^2(\mu)$ if and only if

$$\int \frac{\log h(t)}{t^2+1}\, dt = -\infty$$

and from here it is easy to obtain Theorem 1.1, see, e.g., [1, p. 84]. More recent proofs are exposed in [9, pp. 128, 142; and 4]. If the density h is zero on, say, some compact interval then Krein's condition is not satisfied. The natural question arises if it, in Krein's theorem, suffices to assume convergence of the logarithmic integral of h over the complement of a compact subset of the real line. This is indeed the case: in Section 2 we show that it is sufficient to assume convergence of the logarithmic integral over some subset of so-called positive lower uniform density. The proof is based on an estimate of harmonic measure due to Carleson and on results of Koosis on weighted approximation over subsets of this form. The extension of Krein's result can be regarded as an L^1-version of Koosis' results. However, this version does not seem too well known and I could not find any literature on extensions of this kind.

In Section 3 we give an analogue of Krein's theorem for discrete measures concentrated on the integers; we show that such a measure is indeterminate provided that the logarithmic sum of the masses at the integers converges. This is based on a theorem of Koosis about polynomials.

It is known that if $h(t)\, dt$ is indeterminate where h is an even function and $-\log h(t)$ is a convex function of $\log t$ for $t > 0$ then the logarithmic integral of h is finite (see e.g., [9, p. 170], where a similar result about weighted approximation is given). In general, however, Krein's condition is not a necessary one, as elementary examples show. Using Theorem 2.2 below it is almost trivial to construct, e.g., strictly positive symmetric densities defining indeterminate measures and having infinite logarithmic integrals over the whole real line.

2. A GENERALIZATION

A set $E \subset \mathbf{R}$ is said to have positive lower uniform density if it contains a set of the form

$$E' = \bigcup_{n \in \mathbf{Z}} [a_n - \delta_n, a_n + \delta_n],$$

where the intervals are disjoint and where there exist four constants A, B, δ, Δ such that

$$0 < A < a_n - a_{n-1} < B, \qquad 0 < \delta < \delta_n < \Delta.$$

It is possible to obtain results on weighted approximation over sets E of this form, similar to those of the classical theory; see [9, p. 428]. These results are based on an estimate of harmonic measure for the slit region $\mathscr{D} = \mathbf{C} \setminus E'$ obtained by Carleson (see [7]). See also [9, p. 394; 3]. In our exposition we shall use the following inequality, set as an exercise in [9, pp. 432–434].

THEOREM 2.1. *Suppose that E has positive lower uniform density. There exists a constant C, only depending on E, such that*

$$\int \frac{\log(|p(x)|^2 + 1)}{x^2 + 1} \, dx \leqslant C \int_E \frac{\log(|p(x)|^2 + 1)}{x^2 + 1} \, dx$$

for all polynomials p.

The promised extension of Krein's theorem is an easy corollary to this theorem.

THEOREM 2.2. *If $d\mu(t) = h(t)\, dt \in \mathscr{M}$ and*

$$\int_E \frac{\log h(t)}{t^2 + 1} \, dt > -\infty$$

for some set E of positive lower uniform density then the polynomials are not dense in $L^1(\mu)$. In particular μ is indeterminate.

Proof. We pick an open interval I of E and define l as the characteristic function or indicator function on I. We also pick another open interval J of E, disjoint from I.

Suppose that there were a sequence of polynomials $\{p_k\}$ tending to l in $L^1(\mu)$.

Then, in particular,

$$\int (|p_k(t)|+1)\, h(t)\, dt \leqslant \text{Const}$$

for all k. We take now a constant $c>0$ such that

$$\frac{c}{t^2+1}\, dt$$

is a probability measure on E. By Jensen's inequality we thus obtain

$$\begin{aligned}\text{Const} &\geqslant \log \int (|p_k(t)|+1)\, h(t)\, dt \\ &\geqslant \log \left\{ \frac{1}{c} \int_E (|p_k(t)|+1)\, h(t)\, \frac{c}{t^2+1}\, dt \right\} \\ &\geqslant c \int_E \frac{\log(|p_k(t)|+1)+\log h(t)}{t^2+1}\, dt - \log c.\end{aligned}$$

This means that

$$\int_E \frac{\log(|p_k(t)|^2+1)}{t^2+1}\, dt \leqslant 2 \int_E \frac{\log(|p_k(t)|+1)}{t^2+1}\, dt \leqslant \text{Const}.$$

Theorem 2.1 now implies that

$$\int \frac{\log^+ |p_k(t)|}{t^2+1}\, dt \leqslant 1/2 \int \frac{\log(|p_k(t)|^2+1)}{t^2+1}\, dt \leqslant \text{Const}.$$

for all k. Therefore $\{p_k\}$ is a normal family in the whole complex plane (see, e.g., [12, Proposition 2.1]). A certain subsequence thus converges uniformly over compact subsets to some entire function f, and it follows that $f = l$ μ-a.e. on the real line. This means in turn that the two open sets

$$S_1 = \{t \in I \mid f(t) \neq 1\},$$

$$S_2 = \{t \in J \mid f(t) \neq 0\}$$

are both μ null sets. Therefore $h=0$ a.e. on $S_1 \cup S_2$, so that $\log h = -\infty$ a.e. on $S_1 \cup S_2$. Since $S_1 \cup S_2 \subseteq E$ and h has finite logarithmic integral over E, S_1 and S_2 must have Lebesgue measure zero. Hence $S_1 = S_2 = \varnothing$ and $f \equiv 1$ on I and $f \equiv 0$ on J. This is a contradiction, f being analytic. Therefore no such sequence $\{p_k\}$ exists and the proof is finished.

DEFINITION 2.3. An entire function f is said to belong to the Cartwright class if f is of exponential type and if its logarithmic integral is finite:

$$\int \frac{\log^+|f(t)|}{t^2+1}\,dt<\infty.$$

By a slight modification of the proof above we can show that any function g in the $L^1(\mu)$-closure of the polynomials coincides on E Lebesgue-a.e. with an entire function of Cartwright class: from the assumption $p_k \to_k g$ in $L^1(\mu)$ we obtain that the logarithmic integrals of the p_k's remain bounded. From Proposition 2.1 in [12] we see that $\{p_k\}$ forms a normal family and therefore that a certain subsequence $\{p_{k_l}\}$ converges uniformly over compact subsets to some entire function f of exponential type. Furthermore, by Fatou's lemma,

$$\int \frac{\log^+|f(t)|}{t^2+1}\,dt \leqslant \liminf_l \int \frac{\log^+|p_{k_l}(t)|}{t^2+1}\,dt<\infty.$$

The type of f is actually zero. We have shown that the polynomials are not dense in $L^1(\mu)$. This is well known to be equivalent to

$$\int \frac{\log H(t)}{t^2+1}\,dt<\infty,$$

where H is given as

$$H(z)=\sup\{|p(z)| \mid p \text{ polynomial and } \|p\|_1\leqslant 1\}, \tag{1}$$

see, e.g., [9, p. 158]. In this case, every function in the closure of the polynomials coincides, on the set where $h>0$, a.e. with an entire function of zero exponential type (see [9, p. 160]).

A result similar to Theorem 2.2 holds for sets E of a different structure. We mention a result of Benedicks about weighted approximation over subsets of the form

$$E^p=\bigcup_{n\in\mathbf{Z}} E_n^p,$$

where $E_n^p=[n^p-\delta, n^p+\delta]$, $n\geqslant 0$, $E_{-n}^p=E_n^p$, $n\geqslant 0$, and where the E_n^p's are disjoint. Here $p>1$ and $\delta>0$ are constants. In this situation there is a constant $C>0$, depending only on p and δ such that

$$\int \frac{\log(|p(x)|^2+1)}{x^2+1}\,dx \leqslant C\int_{E^p} \frac{\log(|p(x)|^2+1)}{|x|^{1+1/p}+1}\,dx$$

for all polynomials p. The proof of this is indicated in [9, pp. 428–434, 444–445]. From it we deduce, with a proof almost exactly like the proof of Theorem 2.2, the following result:

THEOREM 2.4. *Suppose $d\mu(t) = h(t)\,dt \in \mathscr{M}$ and that, for some E^p of the above form,*

$$\int_{E^p} \frac{\log h(t)}{|t|^{1+1/p}+1}\,dt > -\infty.$$

Then the polynomials are not dense in $L^1(\mu)$.

We shall use this result to give an example of a symmetric and everywhere positive density h on the real line such that $d\mu(t) = h(t)\,dt$ is indeterminate and yet

$$\int_E \frac{\log h(t)}{t^2+1}\,dt = -\infty$$

for all sets E of positive lower uniform density.

We take, for given $p > 1$, a small $\delta > 0$ and form the set E^p. Then we define

$$h(t) = \begin{cases} e^{-|t|^{1/2p}}, & t \in E^p, \\ e^{-|t|}, & t \in \mathbf{R} \setminus E^p. \end{cases}$$

The condition in Theorem 2.4 is easily seen to be satisfied and therefore $h(t)\,dt$ is indeterminate. Now let E be any set of positive lower uniform density. We have

$$\int_E \frac{-\log h(t)}{t^2+1}\,dt \geqslant \int_{E\setminus E^p} \frac{-\log h(t)}{t^2+1}\,dt = \int_{E\setminus E^p} \frac{|t|}{t^2+1}\,dt,$$

and this integral diverges. Indeed

$$\int_{E\cap E^p} \frac{|t|}{t^2+1}\,dt \leqslant \int_{E^p} \frac{|t|}{t^2+1}\,dt < \infty$$

and

$$\int_E \frac{|t|}{t^2+1}\,dt = \infty,$$

E having positive lower uniform density.

3. A DISCRETE ANALOGUE

Below we give a discrete version of Krein's result. This is based on the following theorem about polynomials, obtained by Koosis.

THEOREM 3.1. *There exist positive constants* η_0 *and* κ *such that for any* $\eta \leqslant \eta_0$ *there is* $C_\eta > 0$ *with the property that*

$$|p(z)| \leqslant C_\eta e^{\kappa\eta\,|z|}, \qquad z \in \mathbf{C},$$

for all polynomials p satisfying

$$\sum \frac{\log^+ |p(n)|}{n^2+1} \leqslant \eta.$$

Koosis proved this result for polynomials of special form in [8] and later for general polynomials, see [9, Chap. VII, Sect. B]. Recently another proof was found, see [12]. As a corollary to the result about polynomials one can obtain a discrete analogue of Akhiezer's theorem on the density question for polynomials in weighted spaces of continuous functions, see [9, p. 523]. Knowing that result it is not too surprising that a discrete analogue of Krein's theorem must hold.

THEOREM 3.2. *Let* $\mu = \sum b_n \varepsilon_n \in \mathscr{M}$ *and suppose that*

$$\sum \frac{\log b_n}{n^2+1} > -\infty.$$

If $f \in L^p(\mu)$, $1 \leqslant p < \infty$, *is in the closure of the polynomials then there is an entire function F of exponential type zero and of Cartwright class such that* $F(n) = f(n)$ *for all integers n.*

Proof. This is inspired by an argument going back to Koosis (see, e.g., [9, p. 523]). Suppose that f can be approximated in $L^p(\mu)$ by polynomials. Then we have a sequence $\{p_k\}$ of polynomials such that $\|f - p_k\|_p \to_k 0$. In particular there is a constant $A > 0$ such that

$$\sum b_n\, |p_k(n)|^p \leqslant A$$

for all k. Therefore $\log^+ |p_k(n)| \leqslant (\log A - \log b_n)/p$ for all n and k. Since $|p_k(n)| \to_k |f(n)|$ for all integers n, Lebesgue's theorem on dominated convergence yields

$$\sum \frac{\log^+ |p_k(n)/m|}{n^2+1} \xrightarrow[k]{} \sum \frac{\log^+ |f(n)/m|}{n^2+1}$$

for any $m \geqslant 1$. Dominated convergence (again) implies that

$$\sum \frac{\log^+ |f(n)/m|}{n^2+1} \xrightarrow[m]{} 0.$$

We want to apply the result about polynomials and in the following we use the constants κ and η_0 from that theorem. Given $\eta \in (0, \eta_0)$ we choose and fix $m \geqslant 1$ such that

$$\sum \frac{\log^+ |f(n)/m|}{n^2+1} \leqslant \eta/2.$$

Therefore we can find $k_0 \geqslant 1$ such that

$$\sum \frac{\log^+ |p_k(n)/m|}{n^2+1} \leqslant \eta.$$

for all $k \geqslant k_0$. By the theorem above there is a constant $C_\eta > 0$ such that

$$|p_k(z)| \leqslant C_\eta m e^{\kappa\eta\, |z|}, \qquad z \in \mathbf{C},$$

for all $k \geqslant k_0$. Therefore $\{p_k\}$ forms a normal family in the whole complex plane. First of all a certain subsequence $\{p_{k_l}\}$ thus converges uniformly over compact subsets to some entire function F. We claim that F is of exponential type zero and of Cartwright class. To any given $\varepsilon > 0$ we choose $\eta \in (0, \eta_0)$ so small that $\kappa\eta \leqslant \varepsilon$. Then we choose $m \geqslant 1$ such that

$$\sum \frac{\log^+ |f(n)/m|}{n^2+1} \leqslant \eta/2.$$

and then finally $l_0 \geqslant 1$ such that for all $l \geqslant l_0$,

$$\sum \frac{\log^+ |p_{k_l}(n)/m|}{n^2+1} \leqslant \eta.$$

Therefore, by the theorem,

$$|p_{k_l}(z)| \leqslant m C_\eta e^{\kappa\eta\, |z|} \leqslant m C_\eta e^{\varepsilon\, |z|}$$

for all $l \geqslant l_0$. This gives $|F(z)| \leqslant m C_\eta e^{\varepsilon\, |z|}$. Since $\varepsilon > 0$ was arbitrary, F is of zero type. It is furthermore of Cartwright class because it has a finite logarithmic sum (see [12, Theorem 1.6]). Finally we note that in fact the original sequence $\{p_k\}$ tends to F u.c.c. To see this it is enough to verify that if two subsequences $\{p_k\}$ and $\{p_{k_s}\}$ converge uniformly over compact subsets to, say, F_1 and F_2 then $F_1 \equiv F_2$. Since $p_k(n) \to_k f(n)$ for all n we have that $F_1(n) - F_2(n) = 0$. Both F_1 and F_2 are of small exponential type (actually of type zero) so from a simple corollary to Jensen's formula (see [9, p. 5]) we get that $F_1 \equiv F_2$. This completes the proof.

Remark 3.3. For $0 < p < 1$, $L^p(\mu)$ is a topological vector space with translation invariant metric given by $d(f, g) = \int |f(t) - g(t)|^p \, d\mu(t)$. The theorem above holds for these values of p as well, with the same proof.

COROLLARY 3.4. *If* $\mu = \sum b_n \varepsilon_n \in \mathscr{M}$ *and*

$$\sum \frac{\log b_n}{n^2+1} > -\infty$$

then the polynomials are not dense in $L^1(\mu)$. *In particular* μ *is indeterminate.*

Proof. It is enough to find $f \in L^1(\mu)$ that is not restriction of an entire function of exponential type zero. We may take f as 1 at the origin and 0 at all other integers. Actually any non-constant bounded function on the integers will do.

Remark 3.5. One may arrive at Theorem 3.2 in a slightly different way. One could first establish Corollary 3.4 by proving that the indicator function of $\{0\}$ is not in the closure of the polynomials and then work with the function H (see (1)), as in [9, Chap. VI, Sect. B.2]. Thereby any function in the closure is restriction of an entire function of zero exponential type. That this entire function has finite logarithmic integral can be seen by Fatou's lemma. In the proof of Theorem 3.2 we did not make use of the function H.

We mention a generalization of the result above to a wider class of discrete measures. Let $h > 0$. A sequence Λ of real numbers is called relatively h-dense in $\mathbf{R}$ if, outside a bounded set, there is at least one point of Λ in any closed interval of length h.

In [13] it is shown how to generalize the result about polynomials working instead with the logarithmic sum over symmetric and relatively h-dense sequences Λ,

$$\sum_{\lambda \in \Lambda} \frac{\log^+ |p(\lambda)|}{\lambda^2+1}.$$

Based on this result one can obtain the following

THEOREM 3.6. *Let Λ be symmetric and relatively h-dense in $\mathbf{R}$ for some $h>0$. If $\mu=\sum_{\lambda\in\Lambda} b_\lambda \varepsilon_\lambda \in \mathscr{M}$ satisfies*

$$\sum_{\lambda\in\Lambda} \frac{\log b_\lambda}{\lambda^2+1} > -\infty$$

then the polynomials are not dense in any of the spaces $L^p(\mu)$, $p\in(0,\infty)$. In particular μ is indeterminate.

Theorem 3.6 has a simple extension to discrete measures concentrated on

$$\Lambda^{(p)}=\{|\lambda|^p \operatorname{sign}(\lambda) \mid \lambda\in\Lambda\},$$

where p is a positive integer and where Λ is a symmetric and relatively h-dense sequence on the real line: if $\mu=\sum b_\lambda \varepsilon_{|\lambda|^p \operatorname{sign}(\lambda)} \in \mathscr{M}$ and $\sum(\log b_\lambda)/(\lambda^2+1) > -\infty$ then μ is indeterminate. The proof of this extension is immediate when p is odd. When p is even a reduction to the case of even functions on the positive half-line is necessary. See Lemma 5.1 in [13].

With this in mind it is easy to construct a symmetric and indeterminate measure $\mu=\sum b_n\varepsilon_n\in\mathscr{M}$ where all the b_n's are positive and where

$$\sum \frac{\log b_n}{n^2+1} = -\infty.$$

It suffices to take an odd integer $p>1$ and a symmetric sequence $\{b_n\}$ of positive numbers satisfying $\mu=\sum b_n\varepsilon_n\in\mathscr{M}$, $\sum(\log b_{n^p})/(n^2+1) > -\infty$, and yet $\sum(\log b_n)(n^2+1) = -\infty$.

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